A Specht Filtration of the Permutation Module over KLR Algebras

Tao Qin

University of Sydney

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KLR Algebras

Definition [KL09, Rou08]

Fix $\alpha \in Q^+$ such that $\mathrm{ht}(\alpha) = n$, the KLR algebra R_α is a **k**-algebra with generators: $e(\mathbf{i})$,y_k,ψ_j subject to the following relations:($1 \leq k \leq n$, $1 \leq j < n$ and $\mathbf{i} \in I^\alpha$)

$$e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{i},\mathbf{j}}e(\mathbf{i}), \sum_{\mathbf{i}\in I^\alpha} e(\mathbf{i}) = 1 \tag{1}$$

$$y_r e(\mathbf{i}) = e(\mathbf{i})y_r, \psi_r e(\mathbf{i}) = e(\sigma_r \mathbf{i})\psi_r \tag{2}$$

$$y_r y_s = y_s y_r, \psi_r \psi_s = \psi_s \psi_r \quad \text{if } |r-s| > 1 \tag{3}$$

$$\psi_r y_s = y_s \psi_r \quad \text{if } s \neq r, r+1 \tag{4}$$

$$\psi_r y_{r+1} e(\mathbf{i}) = (y_r \psi_r + \delta_{\mathbf{i}_r,\mathbf{i}_{r+1}})e(\mathbf{i}) \tag{5}$$

$$y_{r+1}\psi_r e(\mathbf{i}) = (\psi_r y_r + \delta_{\mathbf{i}_r,\mathbf{i}_{r+1}})e(\mathbf{i}) \tag{6}$$

$$\psi_r^2 e(\mathbf{i}) = Q_{\mathbf{i}_r,\mathbf{i}_{r+1}}(y_r, y_{r+1})e(\mathbf{i}) \tag{7}$$

$$\psi_r\psi_{r+1}\psi_r e(\mathbf{i}) = \psi_{r+1}\psi_r\psi_{r+1} e(\mathbf{i}) + Q_{\mathbf{i}_r,\mathbf{i}_{r+1},\mathbf{i}_{r+2}}(y_r, y_{r+1}, y_{r+2})e(\mathbf{i}) \tag{8}$$

Cyclotomic KLR algebra

Given $\Lambda \in P^+$, the *cyclotomic KLR algebra* R_α^Λ is defined as the quotient of R_α by the relations

$$y_1^{\langle \Lambda, \alpha_{\mathring{\imath}_1} \rangle} e(\mathring{\imath}) = 0, \quad \mathring{\imath} \in I^\alpha.$$

Important Results

- KLR algebras categorify the negative part of quantum groups. [KL09, Rou08]
- Cyclotomic KLR algebras categorify highest weight modules $V(\Lambda)$. [KK12]
- In type $\mathrm{A}^{(1)}_{e-1}$, cyclotomic KLR algebras are isomorphic to Ariki–Koike algebras. [BK09]
- Cyclotomic KLR algebras of type $\mathrm{A}^{(1)}_{e-1}$ are graded cellular. [HM10]
- ...and many more.

Permutation Modules

Fix a dominant weight Λ and a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$. For each row of length λ_i, there is a unique one-dimensional irreducible R_{α_i}-module, denoted L_i, where $\alpha_i = \mathrm{cont}(\lambda_i)$. The outer tensor product

$$L_1 \boxtimes L_2 \boxtimes \cdots \boxtimes L_r$$

is a module over $R_{\alpha_1} \otimes R_{\alpha_2} \otimes \cdots \otimes R_{\alpha_r}$. Since R_α is a free over this subalgebra, we define the *permutation module* M^λ by

$$M^\lambda := \mathrm{Ind}_{R_{\alpha_1} \otimes \cdots \otimes R_{\alpha_r}}^{R_\alpha} \left(L_1 \boxtimes L_2 \boxtimes \cdots \boxtimes L_r \right).$$

Theorem [KMR12]

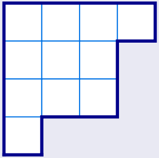
The module M^λ has a **k**-basis indexed by all row-standard tableaux of shape λ.

Specht Modules

Theorem [KMR12]

The Specht module S^λ over the cyclotomic KLR algebra R^Λ_α is isomorphic to the quotient of the permutation module M^λ by the Garnir relations.

Example

Let $\lambda = (4,3,3,1)$ with Young diagram  .

Choose the Garnir node $A = (2,2)$. Then the Garnir tableau G^A is

1	4	5	6
2	3	7	
8	9	10	
11			

.

Universal Construction

Theorem [KMR12]

The Specht module S^λ is isomorphic to $(R_\alpha/J^\lambda)\langle \deg T^\lambda \rangle$ where J^λ is generated by the following relations:

- $e(\mathring{\mathbb{j}}) - \delta_{\mathring{\mathbb{j}}, \mathring{\mathbb{i}}^\lambda}$ for all $\mathring{\mathbb{j}} \in I^\alpha$.
- y_r for $r = 1, \ldots, n$.
- ψ_r whenever r and $r+1$ appear in the same row of T^λ.
- g^A for each Garnir node $A \in [\lambda]$.

Remark

In our case, $g^A = \psi^{G^A}$ for each Garnir node A and Garnir tableau G^A.

Specht Filtration?

We seek a filtration

$$M^\lambda = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_k \supsetneq M_{k+1} = 0$$

such that each quotient M_i/M_{i+1} is isomorphic to a Specht module S^{μ_i} (as an R_α-module) with $\mu_0 = \lambda$.

Warning

Such a filtration does not always exist!

Generalized Specht Filtration in Type A_∞

Let $\lambda = (\lambda_1, \dots, \lambda_k)$ and, for each $1 \le i \le k-1$, fix a Garnir relation ψ^{A_i} between rows i and $i+1$.

Theorem [Q.25]

For $1 \le i \le k-1$, let M_i be the R_α-submodule of M^λ generated by

$$\langle\, \psi^{A_i}, \dots, \psi^{A_{k-1}} \,\rangle.$$

Then

$$M^\lambda = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_{k-1} \supsetneq M_k = 0$$

is a filtration such that for each $1 \le i \le k-1$, there is an exact sequence

$$0 \to S^{\mu_{i,k_i}} \to \cdots \to S^{\mu_{i,1}} \twoheadrightarrow M_i/M_{i+1} \to 0,$$

where each $S^{\mu_{i,j}}$ is a Specht module over $R_\alpha^{\Lambda(i)}$ and $\mu_{i,j} \trianglerighteq \mu_{i,j+1}$ for all admissible i, j.

Hook Partition Case in Type $\mathbf{A}_{e-1}^{(1)}$

In the hook partition case, the same construction yields a Specht filtration.

Example

Let $\lambda = (4, 1^3)$ and $\Lambda = \Lambda_0$ in type $\mathrm{A}_5^{(1)}$. Then there is a Specht filtration

$$M^\lambda = M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq M_3 \supsetneq M_4 \supsetneq 0$$

with

$$S^{\lambda_i} \cong M_i / M_{i+1}, \quad \lambda_0 = \lambda, \ldots, \lambda_4,$$

where $\lambda = \lambda_0$: $\begin{ytableau} 0 & 1 & 2 & 3 \\ 4 \\ 3 \\ 2 \end{ytableau}$ $\quad \lambda_1$: $\begin{ytableau} 4 & 0 & 1 & 2 & 3 \\ 3 \\ 2 \end{ytableau}$

λ_2: $\left(\begin{ytableau} 0 & 1 & 2 & 3 \end{ytableau} \;\middle|\; \begin{ytableau} 3 & 4 \\ 2 \end{ytableau} \right)$ λ_4: $\left(\begin{ytableau} 0 & 1 & 2 & 3 \end{ytableau} \;\middle|\; \begin{ytableau} 4 \end{ytableau} \;\middle|\; \begin{ytableau} 2 & 3 \end{ytableau} \right)$

Thank you!

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